and

$$\sin\left[\frac{4\pi n l_0}{v} f_0\left(1 \mp \frac{1}{2Q}\right)\right] = \mp \sin\frac{n\pi}{Q} = \mp \sin 2x$$

and

$$2nl_0 k(f_Q) = \frac{2\pi l_0}{v}\,\frac{n}{Q} f_0\left(1 \mp \frac{1}{2Q}\right) = \frac{n\pi}{Q}\left(1 \mp \frac{1}{2Q}\right) = 2y$$

$$|i_1| = \frac{|V_1|}{|Z_0|}\,\frac{\cos 2x + \cosh 2y}{\sqrt{(\sin^2 2x + \sinh^2 2y)}}$$

which is the same function as was obtained from the "line length variation method." Similarly, it can be shown that when the two probes are at opposite ends the result is identical, irrespective of whether the line length or the frequency has been changed to obtain the value of the $Q$ factor.

The reason for this is that $|i|_Q/|i|_{max} = f(x)$ where $x = n\pi/(2Q)$ (nepers) which means that the correction to be used in the $Q$-factor measurement is a function of the total attenuation only, and therefore the same result will be obtained by either method of measurement.

# A Nonuniform Coaxial Line with an Isoperimetric Sheath Deformation*

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*Summary*—**For impedance matching in transmission lines, nonuniform lines, obeying laws of taper like the exponential, the Dolph-Chebyshev etc., are used. For the nonuniform coaxial line, constructional advantages can be derived for the same electrical performance if it has a uniform circular inner conductor with an outer conductor having an isoperimetric transition, from circular to elliptic cross section, in conformity with the established laws of taper. This problem has been examined in the paper, and the required design formulas as well as the design charts are developed. The effect of an impedance and geometric discontinuity at the low-impedance junction of such a nonuniform line and the concentric circular uniform line is discussed. The use of the isoperimetric transition line in microwave components is indicated.**

## I. Introduction

A COMMON PROBLEM of systems design is impedance matching. In coaxial transmission lines, this presents difficulties associated with electrical and mechanical design considerations. A solution to the mechanical aspect of the problem has been attempted by the use of nonuniform transmission lines, with a variation of the diameter of either the inner or the outer conductor satisfying the electrical requirements. The latter approach is suggestive of an alternative method wherein the outer conductor transforms isoperimetrically from a circular cross section to an elliptic cross section, the inner conductor being uniformly circular. Mechanical advantages can be derived for the same electrical performance if the increase of ellipticity is related to the existing laws of transition. The proposed structure permits easy and continuous installation of any law of taper. Thus, for instance, the difficulty encountered to install the exponential law[1] or the Orlov's law[2] of taper is not appreciably more than that for a linear taper. The design can also be used for tapered terminations at microwaves,[3] and in the design of microwave components where work on the use of nonuniform lines has been reported.[4,5]

To develop the required design formulas it is first necessary to carry out the field analysis of infinitely long uniform lines with a circular cylindrical inner conductor surrounded by an elliptic outer conductor. Morse and Feshbach[6] give an expression for the case of an inner conductor in the form of a thin wire. A similar analysis can also be made using a Schwarz's transformation.[7] The requirement of the design considered in this paper being that of an inner conductor whose radius is comparable to the dimensions of the ellipse, a different

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[1] C. R. Burrows, "The exponential transmission line," *Bell Sys. Tech. J.*, vol. 17, pp. 555–573; October, 1938.

[2] S. I. Orlov, "Concerning the theory of non-uniform transmission lines," *J. Tech. Phys. USSR*, vol. 26, pp. 2361–2372; October, 1956. (Translated by APS, vol. 1, pp. 2284–2294; October, 1957.)

[3] G. T. Clemens, "A tapered line termination at microwaves," *Quart. J. Appl. Math.*, vol. 7, pp. 425–432; January, 1950.

[4] C. P. Womack, "The use of exponential transmission lines in microwave components," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, pp. 124–132; March, 1962.

[5] R. N. Ghose, "Exponential transmission lines as resonators and transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 213–217; July, 1957.

[6] P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 1203; 1953.

[7] H. A. Schwarz, "Notizia sulla rappresentazione conforme di un'area ellittica sopra un'area circolare," *Annali di Matematica (II)*, vol. 3, pp. 166–173; 1869.

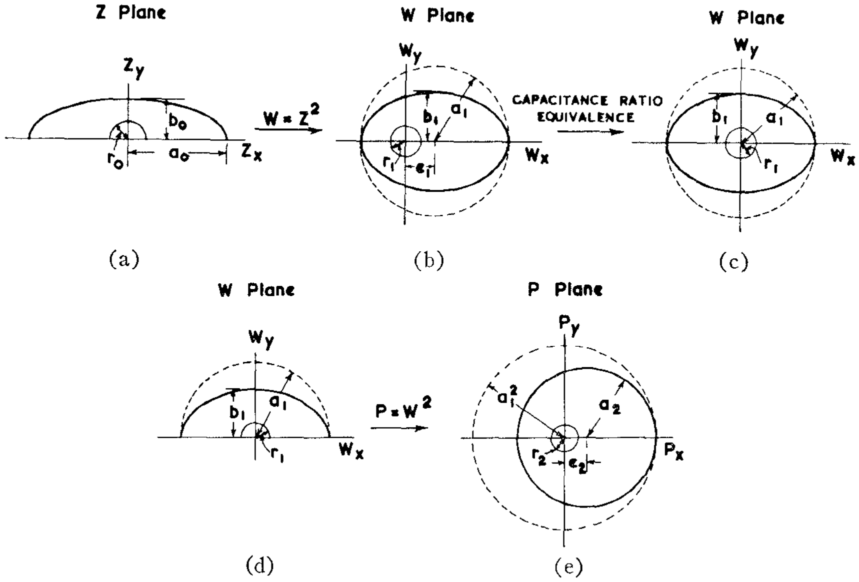


Fig. 1—Sketch showing the transformations and equivalence relations for the determination of the constants of the elliptic-circular line.

approach is made based on conformal transformations, bi-polar coordinates, higher transcendental functions and an impedance ratio equivalence.

The procedure followed in this paper for the actual analysis is as follows. A half section of the infinite elliptic circular uniform line, bifurcated along the major axis of the ellipse, has been conformally transformed into a noncoaxial structure having a circular cylindrical inner conductor with a displaced elliptic sheath of lesser eccentricity than that of the original structure. The capacitance of this transformed structure, and hence that of the original half section, is obtained by a capacitance ratio equivalence, evaluated using the bipolar coordinate and a transformation involving the theta functions and Jacobi's elliptic functions.

A similar method gives the external inductance of the line. Expressions for the characteristic impedance and the phase constant are consequential. Against this background, the design of the nonuniform line is treated as a problem of finding the law of transition, for a given variation of the characteristic impedance so that the electrical performance of existing types of tapered lines, so far as the reflection is concerned, is conserved. Charts are developed for the design of the nonuniform sheath.

## II. Line Constants

A half section of the infinite uniform transmission line with a circular inner conductor and an elliptic sheath having loss free dielectric and perfect conductors is considered.

It is known that a complex plane transformation,[8]

$$W = Z^2, \tag{1}$$

maps the half plane structure in the $Z$ plane given in Fig. 1(a) conformally into the full $W$ plane, noncoaxial in nature. Since only a half plane is considered initially, mapping as on a Riemann surface is not considered; and 1) also maps this region in the manner of a Schlicht function. This structure of the outer conductor in the $W$ plane is an ellipse of lesser eccentricity, Fig. 1(b), than the one initially considered, but there is a deviation in the geometric center in the transformed case given by,

$$e_1 = \tfrac{1}{2}(a_0^2 - b_0^2),$$

where $a_0$ and $b_0$ are respectively the semimajor and the semiminor axes of the elliptic sheath in the $Z$ plane. The resulting structure, though it appears equally difficult to deal with, by the nature of its ellipticity, facilitates a capacitance ratio equivalence. From Figs. 1(b) and 1(c), if

$$\begin{bmatrix} C_{b_1} & C_{c_1} \\ C_{b_1}' & C_{c_1}' \end{bmatrix} = \text{Capacitance between} \begin{bmatrix} \text{eccentric ellipse} & \text{concentric ellipse} \\ \text{eccentric circle} & \text{concentric circle} \end{bmatrix},$$

and the inner conductor whose radius $r_1$ is comparable to the dimension of the ellipse with axes $a_1$ and $b_1$, then, an equivalence

$$\frac{C_{b_1}}{C_{b_1}'} = M_0(a_0/b_0, r_0/b_0)\,\frac{C_{c_1}}{C_{c_1}'}. \tag{2}$$

may be considered. Since the capacitances are constant for a given $a_0$, $b_0$ and $r_0$, $M_0$ is also a constant, but it may vary with $(a_0/b_0)$ and $(r_0/b_0)$. In (2), $C_{b_1}$, the capacitance of the half structure in the $Z$-plane, which is also half the capacitance $C$ of the full original structure, can be found by evaluating $C_{b_1}'$, $C_{c_1}'$, $C_{c_1}$ and $M_0$.

[8] R. V. Churchill, "Complex Variables and Applications," McGraw-Hill Book Co. Inc., New York, N. Y., p. 67; 1960.

The magnitude of $C_{b_1}'$ can be shown, using the bipolar coordinate,[9] to be

$$C_{b_1}' = \frac{2\pi\epsilon}{\text{Cosh}^{-1}\left[\frac{a_0^2b_0^2 + r_1^2}{2a_1r_1}\right]}, \qquad (3)$$

where

$$a_1 = \tfrac{1}{2}(a_0^2 + b_0^2),$$
$$r_1 = r_0^2$$

Also,

$$C_{c_1}' = \frac{2\pi\epsilon}{\ln (a_1/r_1)} \cdot \qquad (4)$$

The capacitance $C_{c_1}$ can be found by a successive transformation,

$$P = W^2, \qquad (5)$$

which up to an axis ratio of about 2.2 transforms the half plane structure of Fig. 1(d) almost into eccentric circular cylinders, the distance between their centers being

$$e_2 = \tfrac{1}{2}(a_1^2 - b_1^2),$$

where $b_1$ is found as follows: Taking the square of the radius vector at an angle of 45° in the $Z$ plane gives

$$e_1, \frac{2a_0^2b_0^2}{a_0^2 + b_0^2}$$

as a point on the ellipse having the semimajor and the semiminor diameters $a_1$ and $b_1$, respectively. Identification of this point in the equation of the ellipse gives

$$b_1 = a_0b_0.$$

Incidentally, it follows that the center of the transformed inner conductor in the $W$-plane is one of the focuses of the ellipse. A similar procedure as adopted for (3) yields

$$C_{c_1} = \frac{4\pi\epsilon}{\text{Cosh}^{-1}\left[\frac{a_1b_1^2 + r_2^2}{2a_2r_2}\right]}, \qquad (6)$$

where, by the properties of the transformation (5),

$$r_2 = r_1^2$$

and

$$a_2 = \tfrac{1}{2}(a_1^2 + b_1^2).$$

$M_0$ in (2) can be evaluated by an approximate method using the Schwarz's transformation involving theta and elliptic functions.[7] A point to decide is the degree of accuracy realized by the approximate method. For this purpose, certain experiments were carried out using the electrolytic tank and they will be described in the following. The index for the capacitance ratios of (2), considering electrode shapes similar to configurations of Fig. 1(b) and 1(c), was taken as the ratio of tank currents. Carefully conducted experiments reveal that for $0<a_0/b_0<3.2$ and $0<r_0/b_0<0.6$, $M_0$ does not deviate from unity by more than about six per cent. But, the above experiments also indicate that $M_0$ is more a function of $(a_0/b_0)$ in the limit $0.1<r_0/b_0<0.6$, than of $(r_0/b_0)$, and hence its value may be determined for the case of an inner conductor in the form a of thin wire of radius corresponding to the lower limit of $(r_0/b_0)$. Thus,

$$\begin{bmatrix} C_{b_2} & C_{c_2} \\ C_{b_2}' & C_{c_2}' \end{bmatrix} = \text{Capacitance between} \begin{bmatrix} \text{eccentric ellipse} & \text{concentric ellipse} \\ \text{eccentric circle} & \text{concentric circle} \end{bmatrix}$$

and the inner conductor in the form of a thin wire of radius $\rho_{i_1}$. Considering a relation similar to (2),

$$\frac{C_{b_2}}{C_{b_2}'} = M_0(a_0/b_0, \rho_{i_0}/b_0)\frac{C_{c_2}}{C_{c_2}'}, \qquad (7)$$

[$\rho_{i_0}$ and $\rho_{i_1}$ are defined by and subsequent to (12)]. But, because of the above consideration implying the approximate independence of $M_0$ with respect to the radius of the inner conductor in the range $0.1<r_0/b_0<0.6$, and also owing to the fact that we are ascertaining only a quantity of a second order magnitude, *viz.*, $M_0$, its value for a given $(a_0/b_0)$ in (2) and (7) are the same to within about two per cent. (It may be restated that this approximation is intended to connect ratios of capacitances only because the actual capacitances are very much functions of $r_0/b_0$.)

In (7) $M_0$ can be evaluated by determining the values of the other parameters. Thus, as of (3),

$$C_{b_2}' = \frac{2\pi\epsilon}{\text{Cosh}^{-1}\left[\frac{a_0^2b_0^2 + \rho_{i_1}^2}{2a_1\rho_{i_1}}\right]}, \qquad (8)$$

and as of (4),

$$C_{c_2}' = \frac{2\pi\epsilon}{\ln (a_1/\rho_{i_1})} \cdot \qquad (9)$$

$C_{c_2}$ and $C_{b_2}$ are evaluated from the transformations,

$$\xi(a_0, b_0) = \sqrt{K_0}\, Sn\left[\frac{2F\left(K_0, \frac{\pi}{2}\right)}{\pi}\sin^{-1}\frac{Z}{\sqrt{a_0^2 - b_0^2}}\right], \qquad (10)$$

[9] Morse and Feshbach,[6] p. 1210.

and

$\xi(a_1, b_1)$

$$= \sqrt{K_1}\, Sn\left[\frac{2F\left(K_1, \frac{\pi}{2}\right)}{\pi}\sin^{-1}\frac{W}{\sqrt{a_1^2 - b_1^2}}\right], \quad (11)$$

where

$$K_0 = \left[\frac{\vartheta_2(0\mid\tau_0)}{\vartheta_3(0\mid\tau_0)}\right]^2$$

$$K_1 = \left[\frac{\vartheta_2(0\mid\tau_1)}{\vartheta_3(0\mid\tau_1)}\right]^2.$$

$\vartheta_2$ and $\vartheta_3$ stand for Theta functions[10] having the Jacobi's parameters,

$$q_0 = \exp(j\pi\tau_0) = \left[\frac{a_0 - b_0}{a_0 + b_0}\right]^2,$$

and

$$q_1 = \exp(j\pi\tau_1) = \left[\frac{a_1 - b_1}{a_1 + b_1}\right]^2.$$

$F(K_0, \pi/2)$ and $F(K_1, \pi/2)$ are Jacobi's elliptic functions of the first kind. Eq. (10) maps the interior of the full structure of the ellipse corresponding to Fig. 1 (a) into the interior of a unit circle. Let us consider

$$\rho_{i_0} = \text{lesser of } (0.1\sqrt{a_0^2 - b_0^2},\ 0.1b_0). \quad (12)$$

The circularity of the inner structure is very nearly preserved under the transformation when it is at the center.

$$G = \frac{\ln\left(\frac{a_1}{\rho_{i_1}}\right)\ln\left(\frac{1}{\rho_{i_0}'}\right)\cosh^{-1}\left(\frac{a_0^2b_0^2 + r_1^2}{2a_1r_1}\right)\cosh^{-1}\left(\frac{a_1^2b_1^2 + r_2^2}{2a_2r_2}\right)}{\ln\left(\frac{a_1}{r_1}\right)\ln\left(\frac{1}{\rho_{i_1}'}\right)\cosh^{-1}\left(\frac{a_0^2b_0^2 + \rho_{i_1}^2}{2a_1\rho_{i_1}}\right)}. \quad (16)$$

However, to obtain more precise information, the geometric mean of the different distances from the center of the transformed inner structure parallel to the $\xi_x$ and $\xi_y$ axes in the $\xi$ plane, can be obtained from the properties of the transformation (10). This mean value is represented by $\rho_{i_0}'$ and is evaluated from the charts and procedure given by Schwarz[7] with the help of tables of transcendental functions.[11] This gives for the transformed structure corresponding to Fig. 1(b),

$$C_{b_2} = \frac{\pi\epsilon}{\ln(1/\rho_{i_0}')}. \quad (13)$$

Similarly, from the transformation (11), and since from (1),

$$\rho_{i_1} = \rho_{i_0}^2,$$

$\rho_{i_1}$ satisfies the requirements of and subsequence to (12) because

$$\rho_{i_1} < \text{lesser of } (0.1\sqrt{a_1^2 - b_1^2},\ 0.1b_1).$$

Thus, (11) applied to the elliptic outer structure and circular inner structure of radius $\rho_{i_1}$, of Fig. 1(c) gives the mean value as $\rho_{i_1}'$, in a manner analogous to that of $\rho_{i_0}'$, so that

$$C_{c_2} = \frac{2\pi\epsilon}{\ln(1/\rho_{i_1}')}. \quad (14)$$

Consequently, since all parameters in (2) except $C_{b_1}$ are known,

$$C_{b_1} = \frac{2\pi\epsilon}{G},$$

and for the full original structure

$$C = \frac{4\pi\epsilon}{G}, \quad (15)$$

where

A further transformation, $S = P^2$ in conjunction with a capacitance ratio equivalence used in the same manner as Figs. 1(b)–1(e), extends the limit of the axis ratio to about 2.8. The procedure can be continued further and the general formula for $G$ to include $N$ such operations, results. Thus,

$$G = \prod_{r=0}^{N}\left[\cosh^{-1}\left(\frac{a_r^2b_r^2 + r^2_{r+1}}{2a_{r+1}r_{r+1}}\right)\right]\prod_{r=0}^{N-1}\left[\frac{\ln\left(\frac{a_{r+1}}{\rho_{i_{r+1}}}\right)\ln\left(\frac{1}{\rho_{i_r}'}\right)}{\ln\left(\frac{a_{r+1}}{r_{r+1}}\right)\ln\left(\frac{1}{\rho_{i_{r+1}}'}\right)\cosh^{-1}\left(\frac{a_r^2b_r^2 + \rho_{i_{r+1}}^2}{2a_{r+1}\rho_{i_{r+1}}}\right)}\right], \quad (17)$$

[10] *Ibid.*, pp. 430, 489.

[11] E. Jahnke and F. Emde, "Tables of Functions with Formulae and Curves," Dover Publications, Inc., New York, N. Y., pp. 41–45, 61–67; 1943.

where

for $r = 1, 2, \cdots, N$,

$$a_r = \tfrac{1}{2}(a_{r-1} + b_{r-1}),$$

$$b_r = (a_{r-1} \cdot b_{r-1}),$$

$$r_r = r_{r-1}^{2},$$

$$\rho_{i_r} = \rho_{i_{r-1}}^{2},$$

and if

$$\rho_{i_0}', \rho_{i_1}' = F_1, F_2(a_0, b_0, \rho_{i_0}),$$

which is evaluated by the procedure following (10) and (11), then

$$\rho_{i_r}', \rho_{i_{r+1}}' = F_1, F_2(a_r, b_r, \rho_{i_r}).$$

The method described above in determining the capacitance is an indirect one and the analysis tends to be approximate only while ascertaining the value of a second order quantity, *viz.*, $M_0$. Thus, the resultant accuracy can be expected to be within about three per cent. But a limitation to the use of (17) as a general formula for $G$ stems from the fact that $M_0$ itself is governed by $a_0/b_0$ and for a ratio greater than 3.2, the desired accuracy to within three per cent is not realized. Thus, $N$ in (17) may be limited to 3.

For $a_0/b_0 > 4$, $0 < r_0/b_0 < 0.6$, it can be seen that the capacitance of the structure depends mostly on $b_0$ owing to its proximity to the inner conductor, and hence the outer conductor can be approximated to parallel planes. Thus, for a similar order of accuracy as considered above, the following known relation[12] may be used in the range $a_0/b_0 > 4$,

$$G = 2 \ln \coth \left[\frac{4b_0}{\pi r_0}\right] (\text{for, } r_0/b_0 < 0.5). \quad (18)$$

Fig. 2 gives the family of curves of $G$ vs $(a_0/b_0)$ with $(r_0/b_0)$ as the parameter. For the range $0.5 < r_0/b_0 < 0.6$ where (18) is not applicable, extrapolation is carried out knowing the trends of the other neighboring curves and they are shown in broken lines.

Similar to (15), it can be shown that the external inductance per unit length

$$1^e = \frac{\mu G}{4\pi}, \quad (19)$$

$\mu$ being the permeability referred to the dielectric between the two conductors. Thus, the secondary constants are

$$Z_0(\text{characteristic impedance}) = \frac{G}{4\pi}\sqrt{\frac{\mu}{\epsilon}}, \quad (20)$$

$$\beta(\text{Phase constant}) = \omega\sqrt{\epsilon\mu}, \quad (21)$$

$\omega$ being the angular frequency.

[12] W. G. Dow, "Fundamentals of Engineering Electronics," John Wiley and Sons, Inc., New York, N. Y., pp. 158–162; 1952.

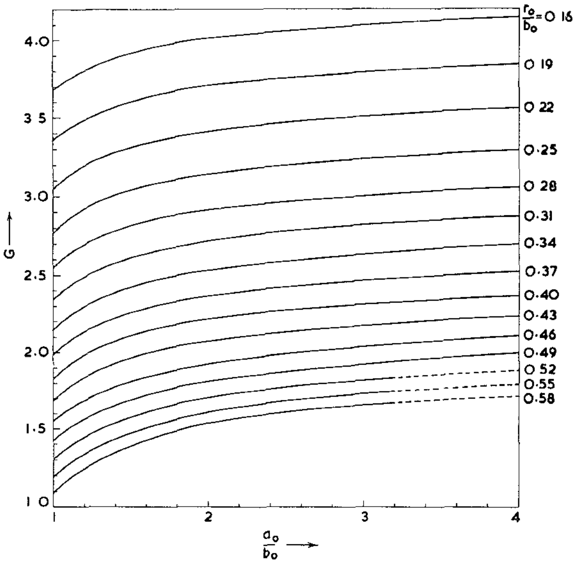


Fig. 2—Curves giving the main parameter of the characteristic impedance $G$ for a given line cross section.

## III. Nonuniform Coaxial Line with Isoperimetric Sheath Transition

The main feature of the uniform line having been established in the preceding section, an extension can now be considered for the nonuniform line.

For the coaxial line, two methods are possible for obtaining gradual and continuous impedance transformation. One way is to taper the inner conductor, while the other is to taper the outer. The methods of realizing the latter type involve a variation of the perimeter of the sheath itself. If an impedance transformer is possible wherein the inner conductor is straight, but the sheath transforms isoperimetrically, mechanical and constructional advantages can be realized. Fig. 3 shows one such method wherein the transition from the circular coaxial to the circular or elliptic-circular coaxial line is isoperimetric. In the following, the electrical design aspects are first considered and a representative design is illustrated. Then the mechanical design considerations, with a view to elucidate some of the constructional advantages, are indicated.

### *Electrical Design Considerations*

Pierce[13] has represented the nonuniform line by a first order nonlinear differential equation in impedance in the Riccati form. From the Fourier pair equations of Bolinder[14] derived from Pierce's equation, it is possible to determine the reflection coefficient as a function of the wavenumber, where the variation of $Z_0$ is known along the line. Thus, for the isoperimetric transition line the electrical performance, so far as the reflection coefficient is concerned, will be the same as for the one whose impedance variation along the tapers is unaltered.

[13] J. R. Pierce, "A note on the transmission line equation in terms of impedance," *Bell Sys. Tech. J.*, vol. 22, pp. 263–265; July, 1943.

[14] E. F. Bolinder, "Fourier transforms and tapered transmission lines," Proc. IRE, (*Correspondence*), vol. 44, p. 557; April, 1956.

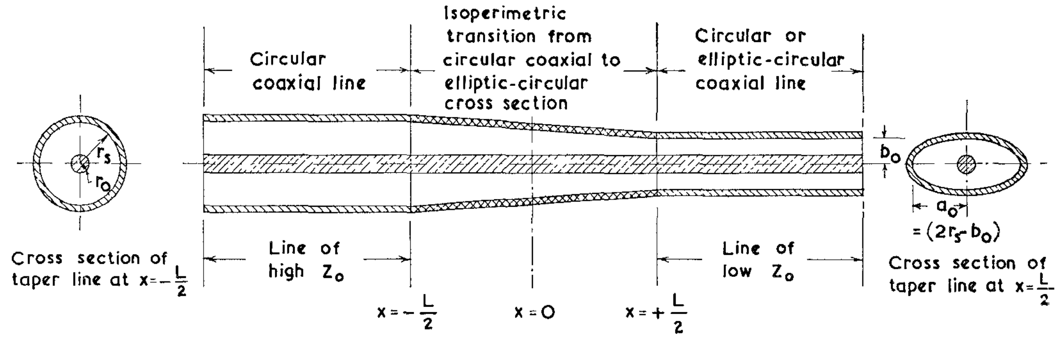


Fig. 3—Sketch representing the proposed structure of the nonuniform line with isoperimetric sheath transition.

Because of this, the impedance relation derived for the elliptic-circular case, the major and minor axis diameters being treated as functions of $x$, (*i.e.*, $a_{0x}$ and $b_{0x}$, respectively), can be equated to the impedance relations derived for the various laws of taper in existing designs.

Table I gives equations for the sheath design for two representative cases, *viz.*, the exponential[15] and the Dolph-Chebyshev.[16] Column two of this table gives the necessary design identities for the transition of the sheath from the circular coaxial high-impedance uniform line to the elliptic-circular or circular coaxial low impedance line to give the already established relations for reflection coefficients as given in column three. Sufficiency of the design equations are obtained from the following constraint resulting from the isoperimetric nature of the transition:

$$a_{0x} = 2r_s - b_{0x}, \tag{22}$$

where $r_s$ is the sheath radius of the high-impedance uniform line. Design equations for the sheath can be similarly obtained for the law resulting from Orlov's nonuniform line synthesis.[2] This law is also derived by Sharpe[17] by treating the analysis of a nonuniform line as a one-dimensional scattering problem by converting the line equations into an integral equation, by using the appropriate Green's function which yields a solution on truncating the Fredholm series. The Orlov's equations that result may be related to the tapers obtained by Willis and Sinha[18] for the matched line and its extension to the general case of arbitrary reflection coefficients by Cath.[19]

A design chart is developed which directly gives the dimensions for the nonuniform sheath. This chart results from the expression for $G$ and (22) and is given in Fig. 4. (Since $a_0/b_0$ cannot be less than unity, there is a forbidden region for the curves which is indicated in the figure.) For illustration, the profile of the outer conductor is designed for the exponential transition from 75Ω to 52Ω. For a particular value of $x$, the right-hand side of equations of Table I, column 2, can be enumerated. This also gives the value of $G(r_s/r_0,\ a_{0x}/b_{0x},\ r_0/b_{0x})$. Knowing $(r_s/r_0)$ from the dimensions of the high impedance uniform line, the value of $(a_0/b_0)$ can be read off the curves of Fig. 4. Knowing $G$ and $(a_0/b_0)$ the value of $(r_0/b_0)$ can be obtained from the curves of Fig. 2. Thus, for a known $r_s$, $r_0$ and $x$, $a_0$ and $b_0$ can be determined. The profile so determined for the exponential line is compared with the profile of a conventional design with a nonuniform inner conductor, of radius $r_i(x)$, and constant sheath radius, $r_s$ (Fig. 5).

*Mechanical Considerations*

To bring about the nature of the mechanical problem that is confronted in the actual fabrication, a representative technique for the construction of the nonuniform line with the isoperimetric sheath deformation will be sketched. It is known that by a rolling process with the surface of revolution of an inverted half ellipse, a plastic deformation from circular hollow cylinder to an isoperimetric elliptic hollow cylinder can be obtained. For the nonuniformity to be installed the pressure on the rollers, with the solid of revolution corresponding to the ellipse of the low impedance end of the nonuniform line, need be varied as a function of $x$. Expressions for the critical pressure of the plastic deformation can be taken as those based on Southwell's.[20] The rollers of the described shape conditions the deformations nearer an ellipse than the plane rollers. For higher ellipticities the parameter of importance deciding the characteristic impedance is $b_0$ and slight deviation from the shape of an ellipse does not cause much change in the impedance. The displacement of one of the rollers, the other being fixed, is directed by the pressure applied to it and the resistance offered by the hollow cylindrical sheath to buckling. Knowing the parameters of the sheath, such as the tangential modulus of the material, the shell

[15] R. E. Collin, "The optimum tapered transmission line matching section," Proc. IRE, vol. 44, pp. 539–548; April, 1956.

[16] R. W. Klopfenstein, "A transmission line taper of improved design," Proc. IRE, vol. 44, pp. 31–35; January, 1956.

[17] C. B. Sharpe, "An alternative derivation of Orlov's synthesis formula for non-uniform lines," *Proc. IEE*, Mono. No. 483 E; November, 1961.

[18] J. Willis and N. K. Sinha, "Non-uniform transmission lines as impedance transformers," *Proc. IEE*, Paper No. 1961 R, vol. 103B, pp. 166–172; March, 1956.

[19] P. G. Cath, "The synthesis of non-uniform transmission lines," Cooley Electronics Lab., Univ. of Michigan, Ann Arbor, Rept. No. 116; January, 1961.

[20] S. Timoshenko, "Strength of Materials," McGraw-Hill Book Co., New York, N. Y., pt. 2, pp. 186–190; 1956.

TABLE I

| Type of transition | Necessary design equation for circular coaxial to elliptic-circular coaxial transition | Reflection coefficient | Expansion of symbols |
|---|---|---|---|
| Exponential[15] | $G(r_s/r_0,\ a_{0x}/b_{0x},\ r_0/b_{0x})$ $= 4\pi\sqrt{\frac{\epsilon}{\mu}}\exp\left[\frac{x}{L}\ln\frac{Z_2}{Z_1}+\frac{1}{2}\ln Z_1Z_2\right]$ | $\frac{1}{2}e^{-j\beta L}\ln\frac{Z_2}{Z_1}\left(\frac{\sin\beta L}{\beta L}\right)$ | $Z_1$ = Impedance at sending end; $Z_2$ = Impedance at receiving end; $L$ = Total length of transition; $x$ = Distance reckoned from center of transition; $\beta$ = Phase constant; $j$ = Complex operator; $A$ = Parameter which determines maximum magnitude of reflection in pass band; $\phi(Z, A)$ given by the tables of Klopfenstein[16]; $U$ = Unit step function; $U(Z) = 0$ if $Z < 0$, $= 1$ if $Z \geq 0$ |
| Dolph-Chebyshev[16] | $G(r_s/r_0,\ a_{0x}/b_{0x},\ r_0/b_{0x})$ $= 4\pi\sqrt{\frac{\epsilon}{\mu}}\exp\left[\frac{1}{2}\ln(Z_1Z_2)+\frac{\rho_0}{\cosh A}\cdot\{A^2\phi(2Z/L, A)+U(x-L/2)-U(-x-L/2)\}\right]$ $\lvert x\rvert \leq L/2$; $= 4\pi Z_2\sqrt{\frac{\epsilon}{\mu}}$ $x > \frac{L}{2}$; $= 4\pi Z_1\sqrt{\frac{\epsilon}{\mu}}$ $x < -\frac{L}{2}$ | $\frac{1}{2}e^{-j\beta L}\ln\frac{Z_2}{Z_1}\left(\frac{\cos\sqrt{(\beta L)^2-A^2}}{\cosh A}\right)$ | |

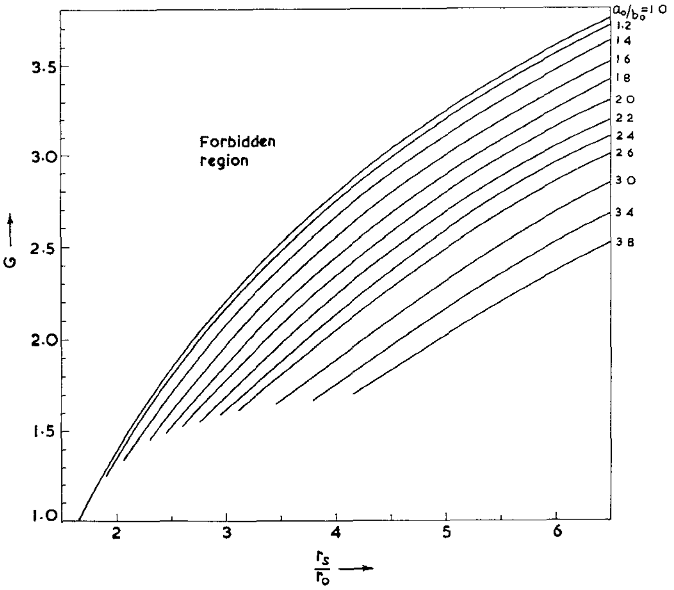

Fig. 4—Chart for the nonuniform sheath design.

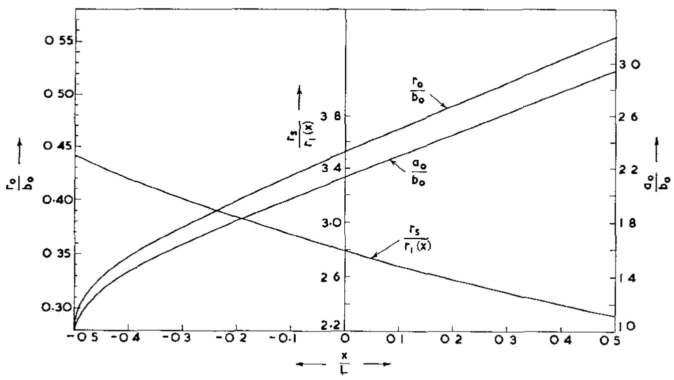

Fig. 5—Comparison of the normalized profile diagrams of the proposed design with an existing type for a 75Ω to 52Ω impedance transformation based on exponential transition.

thickness, the buckling properties and also the law of variation of $b_{0x}$ of the ellipse as a function of the length along the non-uniform line, the pressure on the roller can be programmed. A simple electromechanical control scheme is possible which compares the $b_{0x}$ measured directly on the work with the $b_{0x}$ obtained from the profile diagram similar to that of Fig. 5, and the difference of these two values is the index for correcting the pressure on the rollers.

Some of the more important constructional features of the isoperimetric transition are the following:

1) The deformation of the shape of the sheath can be obtained by a "pressing into shape" operation and hence there is no removal of metal.
2) The inner conductor is uniformly circular throughout the length of the line.
3) Smooth and flawless deformations are possible when $(h/r_s)$ is small, where $h$ is the shell thickness.
4) Any law of the transition involving smooth and continuous variation of the characteristic impedance can be inscribed on it.
5) Sending end and receiving end lines may both be circular coaxial if a discontinuity at the low impedance junction described in the next section is taken care of.

Against these advantages, there are also the following disadvantages:

1) The junction of the uniform circular coaxial line and the nonuniform line at the low impedance end enforce some difficulties because soldered and butt joints are often inadmissible.
2) For higher ellipticities, geometric discontinuity may excite spurious transmission modes.
3) Mounting of the inner conductor by structures like the dielectric beads are not as flexible as in circular coaxial transitions.

## IV. Isoperimetric Line for Impedance Matching in Uniform Lines

When the isoperimetric sheath transition is used for impedance matching in uniform lines, the following feature should be examined. Because of the elliptic nature of the cross section at the low impedance end of the nonuniform line the nature of its junction poses some special considerations. The high impedance uniform line and the high impedance end of the nonuniform line have both the same cross section, *viz.*, concentric circles with the same radius, and hence the junction is perfect. But this is not the case at the low impedance junction which is between an ellipse having the same circumference as the high impedance line sheath and a circle whose circumference is different from that of this ellipse. Two cases are discussed in the following.

1) If a small impedance discontinuity[21] can be tolerated, the uniform line can be circular coaxial having a sheath radius equal to the minor axis of the ellipse. Assuming that the nonuniform line tapers gradually, the term contributing to the reflection may be taken as,

$$\rho_d = \frac{G_{(L/2)} - 2\ln\left[\dfrac{b_{0(L/2)}}{r_0}\right]}{G_{(L/2)} + 2\ln\left[\dfrac{b_{0(L/2)}}{r_0}\right]}. \tag{23}$$

Here $G_{(L/2)}$ and $b_{0(L/2)}$ are the values of $G$ and $b_{0x}$ at the low impedance end of the taper line. Further analysis of the effect of this discontinuity may be treated for the same impedance relations, as for one with a circular sheath of radius

$$\left(r_0 \exp \frac{G_{(L/2)}}{2}\right)$$

stepping down to a radius of $b_{0(L/2)}$. This analysis is then the same as the one dealt with by Marcuvitz.[22] The impedance discontinuity resulting in reflection given by (23) is considerable in many designs, but this may be reduced by alternate means. The low impedance end of the nonuniform line may be so proportioned that the minor axis of the ellipse is less than the radius of the uniform line to such an extent that the characteristic impedances become equal. In this case there is no impedance discontinuity, thus mitigating reflection, but a purely geometric discontinuity[21] is present which when considerable may initiate spurious transmission modes. Taking note of these two requirements, the transition as well as the discontinuities may be optimized. As an example, in the Dolph-Chebyshev taper an impedance discontinuity is introduced by Klopfenstein for a better electrical performance. Hence, the ellipticity of the ellipse at the low impedance end may be proportioned to include this necessary rise in impedance, while the rest of the discontinuity may be geometrical.[21]

2) As a special case the low impedance uniform line itself may be of elliptic-circular cross section in which case the junction is again perfect. The properties of such a line and its merits and demerits over existing types of uniform lines are under examination. The TEM wave equation for such a uniform line is conveyed in a separate communication.[23]

## V. Isoperimetric Line for Microwave Components

The proposed taper line can also be utilized in microwave components with considerable constructional advantage. Some of these applications are sketched below.

1) The exponential transmission line resonator is known to have been employed in coaxial cavities of RF amplifiers.[4,5] Womack observes that exact exponential tapers for coaxial lines are difficult to construct, and hence the advantages of shorter resonator lengths and, consequently, less weight may not fully justify the increased production difficulties. The statement holds good so far as the usual nonisoperimetric transition of either the inner or the outer conductor. For the isoperimetric transition as dealt with here, the constructional difficulties are the same whether it is for a linear taper as dealt with by Womack or the actual exponential taper, and both do not present as much difficulty as the nonisoperimetric transitions. Thus, employing the exponential taper itself, 30 to 40 per cent[24] shorter length can be obtained than the corresponding uniform line resonators at the same frequency, and about 15 per cent shorter than using the linear taper.

2) A complementary case to the preceding arises if a convergent line is considered wherein the same percentage of an increase of length is obtained which is worth reckoning at microwave frequencies where fabrication difficulties may be encountered owing to too short a length of the line.

3) The design can be adopted for tapered line terminations at microwaves in the same way as was done by Clemens[3] for the nonisoperimetric transition but with better constructional advantage as described in section III.

[21] For the purpose of the present paper the following terms are defined or redefined: An impedance discontinuity is an abrupt change in impedance which consequently calls for a change in the size and/or shape of the cross-sectional geometries.

A geometric discontinuity is a change in the size and/or shape of the cross-sectional geometries which may or may not be accompanied by a change in impedance.

A purely geometric discontinuity is a change in the size and/or shape of the cross-sectional geometries without any change in impedance.

[22] N. Marcuvitz, "Wave Guide Handbook," McGraw-Hill Book Company Inc., New York, N. Y., p. 311; 1947.

[23] N. Seshagiri, "A uniform coaxial line with an elliptic-circular cross-section," this issue, page 549.

[24] The values are based on those of Womack.[4]

## Conclusion

It has been shown in the paper that a nonuniform coaxial transmission line having an isoperimetric sheath deformation for impedance matching in uniform lines or for use in microwave components can be designed. It has also been shown that such a configuration can yield an electrical performance corresponding to some of the commonly adopted laws of taper. The new structure is believed to have certain distinct mechanical and constructional advantages. A possible mechanical setup, with its associated control equipment, for the construction of such a line, is suggested. As this setup is only representative, it is expected that further refinements may entail a more economical and accurate construction.

## Acknowledgment

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# Prototypes for Use in Broadbanding Reflection Amplifiers*

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*Summary*—This paper tabulates, as functions of reflection gain and ripple, the element values of negative-resistance terminated, prototype, low-pass, lumped-element ladder networks of normalized impedance and bandwidth. (The values are calculated using known synthesis methods.) Next, it provides a technique for relating the characteristics of any actual narrow-band, negative-resistance device to the value of the prototype susceptive element adjacent to the negative resistance. When an actual negative-resistance device has been related to a prototype in this manner, the performance of the device with one, two or three additional cascaded resonators can be predicted from given graphs. This allows trade-offs among gain, ripple, and bandwidth, within limits. Finally, the predicted performance can be used with simple formulas and the table of prototype element values to design suitable resonators to broadband the actual amplifier. The tables and techniques of this paper are used successfully to broadband tunnel-diode, maser and parametric-amplifier circuits.

This paper allows the practical engineer to estimate the broadbanding potential of any given negative-resistance device and provides him with the proper element values to do so with only a few very simple calculations required.

## Introduction

IN RECENT YEARS reflection-type, negative-resistance amplifiers have received considerable attention. This has come about as a result of the introduction of circulators and of solid-state devices that are capable of presenting a negative resistance under the proper conditions. Typically, such an amplifier might consist of a negative resistance and an associated resonating structure terminating one port of a circulator. Assuming an ideal circulator to which the load and generator resistances are matched, the mid-band gain of the amplifier is determined by the ratio of the load resistance to the negative resistance. The bandwidth of the amplifier depends on the values of these resistances and the slope parameter of the resonating structure. A considerable improvement in the bandwidth of such an amplifier can be achieved by appropriately placing one or more additional resonating structures between the circulator and the terminals of the negative resistance (with its own resonating structure). Matthaei[1] has shown how this can be done for varactor-diode parametric amplifiers, while Kyhl, McFarlane and Strandberg[2] have demonstrated the use of an additional cavity to broaden the bandwidth of the cavity maser. While these references consider broadbanding from the point of view of the specific negative-resistance device employed, this paper considers the broadbanding of a very simple prototype device and discusses how to relate this prototype to any particular negative-resistance device. The results are applicable to many kinds of negative-resistance devices, yet only to the extent that they can be reasonably related to the prototype.

Design relations for broadbanding ideal negative-resistance devices (capacitance and negative-resistance in parallel) have previously been given for both maxi-

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[1] G. L. Matthaei, "A study of the optimum design of wide-band parametric amplifiers and up-converters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-9, pp. 23–38; January, 1961.

[2] R. L. Kyhl, R. A. McFarlane, and M. W. P. Strandberg, "Negative L and C in solid-state masers," Proc. IRE, vol. 50, pp. 1608–1623; July, 1962.